EXTRAS

- SPIN
- ANGULAR
- LADDER

$S_\pm \equiv S_x \pm i S_y$ what does it do to $\eta(3)$?

$S_z \eta(\varepsilon) = \frac{\hbar}{2}\varepsilon \eta(\varepsilon)$

$[S_i, S_j] = i\hbar \varepsilon_{ijk} S_k$ ↑ LEVI-CIVITA

$[S_z, S_\pm] = S_z S_\pm - S_\pm S_z = S_z S_x \pm i S_z S_y - S_x S_z \mp i S_y S_z$

$[S_z, S_x] = i\hbar \varepsilon_{zxy} S_y = +i\hbar S_y$

ε_{xyz}, ε_{yzx}, ε_{zxy}: 1; flip: −1

$\pm i [S_z, S_y] = \pm i\, i\hbar \varepsilon_{zyx} S_x = \pm\hbar S_x$

$= \pm\hbar S_x + i\hbar S_y = \pm\hbar [S_x \pm i S_y]$

$[S_z, S_\pm] = \pm\hbar S_\pm$

what $S_\pm$ does to $\eta(\varepsilon)$? measure it

$S_z [S_\pm \eta(\varepsilon)] = [[S_z S_\pm - S_\pm S_z] + S_\pm S_z]\eta(\varepsilon)$

$= [[S_z S_\pm] + S_\pm S_z]\eta(\varepsilon)$

$= [\pm\hbar S_\pm + S_\pm S_z]\eta(\varepsilon)$

$= [\pm\hbar S_\pm + S_\pm \frac{\hbar}{2}\varepsilon]\eta(\varepsilon)$

$= \hbar[\pm 1 + \frac{1}{2}\varepsilon] S_\pm \eta(\varepsilon)$

S_+ $\varepsilon = 1$ $+1+\frac{1}{2}$ outside space $\Rightarrow 0$ $S_+\eta(+1) = 0$ $S_+\eta(-1) = \hbar\eta(1)$

$\varepsilon = -1$ $+1 - \frac{1}{2} = \frac{\hbar}{2} \to \varepsilon = 1$ $\varepsilon = 1$

S_- when $\varepsilon = 1$ ⟶ $-1 + \frac{1}{2} = -\frac{1}{2} \to -1$ $S_-\eta(+1) = \hbar\eta(-1)$ $S_-\eta(-1) = 0$

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Angular momentum operator

In quantum mechanics, the **angular momentum operator** is one of several related operators analogous to classical angular momentum. The angular momentum operator plays a central role in the theory of atomic physics and other quantum problems involving rotational symmetry. In both classical and quantum mechanical systems, angular momentum (together with linear momentum and energy) is one of the three fundamental properties of motion.[1]

There are several angular momentum operators: **total angular momentum** (usually denoted **J**), **orbital angular momentum** (usually denoted **L**), and **spin angular momentum** (**spin** for short, usually denoted **S**). The term *angular momentum operator* can (confusingly) refer to either the total or the orbital angular momentum. Total angular momentum is always conserved, see Noether's theorem.

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$\bar{p} = m\bar{v} = m\frac{\partial \bar{r}}{\partial t} = $

$\hat{p} = -i\hbar\frac{\partial}{\partial \bar{r}} = -i\hbar\nabla_r$

$\bar{L} = \bar{r}\times\bar{p}$

$\hat{L} = \hat{r}\times\hat{p} = \hat{r}\times -i\hbar\nabla_r$

$\overset{!}{=} -i\hbar(\hat{r}\times\bar{\nabla}_r)$

Overview

In quantum mechanics, angular momentum can refer to one of three different, but related things.

Orbital angular momentum

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The classical definition of angular momentum is $\mathbf{L} = \mathbf{r} \times \mathbf{p}$. The quantum-mechanical counterparts of these objects share the same relationship:

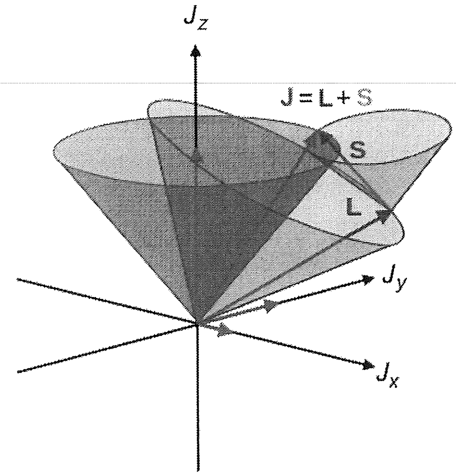
$$\mathbf{L} = \mathbf{r} \times \mathbf{p}$$

where $\mathbf{r}$ is the quantum position operator, $\mathbf{p}$ is the quantum momentum operator, × is cross product, and $\mathbf{L}$ is the *orbital angular momentum operator*. $\mathbf{L}$ (just like $\mathbf{p}$ and $\mathbf{r}$) is a *vector operator* (a vector whose components are operators), i.e. $\mathbf{L} = (L_x, L_y, L_z)$ where L_x, L_y, L_z are three different quantum-mechanical operators.

In the special case of a single particle with no electric charge and no spin, the orbital angular momentum operator can be written in the position basis as:

$$\mathbf{L} = -i\hbar(\mathbf{r} \times \nabla)$$

where ∇ is the vector differential operator, del.



"Vector cones" of total angular momentum **J** (purple), orbital **L** (blue), and spin **S** (green). The cones arise due to quantum uncertainty between measuring angular momentum components (see below).

Spin angular momentum

There is another type of angular momentum, called *spin angular momentum* (more often shortened to *spin*), represented by the spin operator $\mathbf{S}$. Spin is often depicted as a particle literally spinning around an axis, but this is only a metaphor: spin is an intrinsic property of a particle, unrelated to any sort of motion in space. All elementary particles have a characteristic spin, which is usually nonzero. For example, electrons always have "spin 1/2" while photons always have "spin 1" (details below).

Total angular momentum

Finally, there is total angular momentum $\mathbf{J}$, which combines both the spin and orbital angular momentum of a particle or system:

$$\mathbf{J} = \mathbf{L} + \mathbf{S}.$$

Conservation of angular momentum states that $\mathbf{J}$ for a closed system, or $\mathbf{J}$ for the whole universe, is conserved. However, $\mathbf{L}$ and $\mathbf{S}$ are *not* generally conserved. For example, the spin–orbit interaction allows angular momentum to transfer back and forth between $\mathbf{L}$ and $\mathbf{S}$, with the total $\mathbf{J}$ remaining constant.

Commutation relations

Commutation relations between components

The orbital angular momentum operator is a vector operator, meaning it can be written in terms of its vector components $\mathbf{L} = (L_x, L_y, L_z)$. The components have the following commutation relations with each other:[2]

$$[L_x, L_y] = i\hbar L_z, \quad [L_y, L_z] = i\hbar L_x, \quad [L_z, L_x] = i\hbar L_y,$$

where [,] denotes the commutator

$$[X, Y] \equiv XY - YX.$$

This can be written generally as

$$[L_l, L_m] = i\hbar \sum_{n=1}^{3} \varepsilon_{lmn} L_n,$$

where l, m, n are the component indices (1 for x, 2 for y, 3 for z), and ε_{lmn} denotes the Levi-Civita symbol.

A compact expression as one vector equation is also possible:[3]

$$\mathbf{L} \times \mathbf{L} = i\hbar \mathbf{L}$$

The commutation relations can be proved as a direct consequence of the canonical commutation relations $[x_l, p_m] = i\hbar\delta_{lm}$, where δ_{lm} is the Kronecker delta.

There is an analogous relationship in classical physics:[4]

$$\{L_i, L_j\} = \varepsilon_{ijk} L_k$$

where L_n is a component of the *classical* angular momentum operator, and $\{,\}$ is the Poisson bracket.

The same commutation relations apply for the other angular momentum operators (spin and total angular momentum):[5]

$$[S_l, S_m] = i\hbar \sum_{n=1}^{3} \varepsilon_{lmn} S_n, \quad [J_l, J_m] = i\hbar \sum_{n=1}^{3} \varepsilon_{lmn} J_n.$$

These can be *assumed* to hold in analogy with $\mathbf{L}$. Alternatively, they can be *derived* as discussed below.

These commutation relations mean that $\mathbf{L}$ has the mathematical structure of a Lie algebra, and the ε_{lmn} are its structure constants. In this case, the Lie algebra is SU(2) or SO(3) in physics notation ($\mathbf{su}(2)$ or $\mathbf{so}(3)$ respectively in mathematics notation), i.e. Lie algebra associated with rotations in three dimensions. The same is true of $\mathbf{J}$ and $\mathbf{S}$. The reason is discussed below. These commutation relations are relevant for measurement and uncertainty, as discussed further below.

Commutation relations involving vector magnitude

Like any vector, a magnitude can be defined for the orbital angular momentum operator,

$$L^2 \equiv L_x^2 + L_y^2 + L_z^2 \ .$$

L^2 is another quantum operator. It commutes with the components of $\boldsymbol{L}$,

$$\left[L^2, L_x\right] = \left[L^2, L_y\right] = \left[L^2, L_z\right] = 0 \ .$$

One way to prove that these operators commute is to start from the $[L_\ell, L_m]$ commutation relations in the previous section:

Click [show] on the right to see a proof of $[L^2, L_x] = 0$, starting from the $[L_\ell, L_m]$ commutation relations[6]

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$$\begin{aligned}\left[L^2, L_x\right] &= \left[L_x^2, L_x\right] + \left[L_y^2, L_x\right] + \left[L_z^2, L_x\right] \\ &= L_y\left[L_y, L_x\right] + \left[L_y, L_x\right]L_y + L_z\left[L_z, L_x\right] + \left[L_z, L_x\right]L_z \\ &= L_y\left(-i\hbar L_z\right) + \left(-i\hbar L_z\right)L_y + L_z\left(i\hbar L_y\right) + \left(i\hbar L_y\right)L_z \\ &= 0\end{aligned}$$

Mathematically, L^2 is a Casimir invariant of the Lie algebra **SO(3)** spanned by $\mathbf{L}$.

As above, there is an analogous relationship in classical physics:

$$\left\{L^2, L_x\right\} = \left\{L^2, L_y\right\} = \left\{L^2, L_z\right\} = 0$$

where L_i is a component of the *classical* angular momentum operator, and $\{,\}$ is the Poisson bracket.[7]

Returning to the quantum case, the same commutation relations apply to the other angular momentum operators (spin and total angular momentum), as well,

$$\begin{aligned}\left[S^2, S_i\right] &= 0, \\ \left[J^2, J_i\right] &= 0.\end{aligned}$$

Uncertainty principle

In general, in quantum mechanics, when two observable operators do not commute, they are called complementary observables. Two complementary observables cannot be measured simultaneously; instead they satisfy an uncertainty principle. The more accurately one observable is known, the less accurately the other one can be known. Just as there is an uncertainty principle relating position and momentum, there are uncertainty principles for angular momentum.

The Robertson–Schrödinger relation gives the following uncertainty principle:

$$\sigma_{L_x}\sigma_{L_y} \geq \frac{\hbar}{2}\left|\langle L_z\rangle\right|.$$

where σ_X is the standard deviation in the measured values of X and $\langle X\rangle$ denotes the expectation value of X. This inequality is also true if x, y, z are rearranged, or if L is replaced by J or S.

Therefore, two orthogonal components of angular momentum (for example L_x and L_y) are complementary and cannot be simultaneously known or measured, except in special cases such as $L_x = L_y = L_z = 0$.

It is, however, possible to simultaneously measure or specify L^2 and any one component of L; for example, L^2 and L_z. This is often useful, and the values are characterized by the azimuthal quantum number (l) and the magnetic quantum number (m). In this case the quantum state of the system is a simultaneous eigenstate of the operators L^2 and L_z, but *not* of L_x or L_y. The eigenvalues are related to l and m, as shown in the table below.

Quantization

In quantum mechanics, angular momentum is *quantized* – that is, it cannot vary continuously, but only in "quantum leaps" between certain allowed values. For any system, the following restrictions on measurement results apply, where $\hbar$ is reduced Planck constant:

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If you measure...	...the result can be...	Notes
L_z	$(\hbar m)$, where $m \in \{\ldots, -2, -1, 0, 1, 2, \ldots\}$	*m* is sometimes called *magnetic quantum number*. This same quantization rule holds for any component of **L**; e.g., L_x or L_y. This rule is sometimes called **spatial quantization**.[8]
S_z or J_z	$(\hbar m)$, where $m \in \{\ldots, -1, -0.5, 0, 0.5, 1, 1.5, \ldots\}$	For S_z, *m* is sometimes called *spin projection quantum number*. For J_z, *m* is sometimes called *total angular momentum projection quantum number*. This same quantization rule holds for any component of **S** or **J**; e.g., S_x or J_y.
L^2	$(\hbar^2 \ell(\ell+1))$, where $\ell \in \{0, 1, 2, \ldots\}$	L^2 is defined by $L^2 \equiv L_x^2 + L_y^2 + L_z^2$. ℓ is sometimes called *azimuthal quantum number* or *orbital quantum number*.
S^2	$(\hbar^2 s(s+1))$, where $s \in \{0, 0.5, 1, 1.5, \ldots\}$	*s* is called spin quantum number or just *spin*. For example, a spin-½ particle is a particle where *s* = ½.
J^2	$(\hbar^2 j(j+1))$, where $j \in \{0, 0.5, 1, 1.5, \ldots\}$	*j* is sometimes called *total angular momentum quantum number*.
L^2 and L_z simultaneously	$(\hbar^2 \ell(\ell+1))$ for L^2, and $(\hbar m_\ell)$ for L_z where $\ell \in \{0, 1, 2, \ldots\}$ and $m_\ell \in \{-\ell, (-\ell+1), \ldots, (\ell-1), \ell\}$	(See above for terminology.)
S^2 and S_z simultaneously	$(\hbar^2 s(s+1))$ for S^2, and $(\hbar m_s)$ for S_z where $s \in \{0, 0.5, 1, 1.5, \ldots\}$ and $m_s \in \{-s, (-s+1), \ldots, (s-1), s\}$	(See above for terminology.)
J^2 and J_z simultaneously	$(\hbar^2 j(j+1))$ for J^2, and $(\hbar m_j)$ for J_z where $j \in \{0, 0.5, 1, 1.5, \ldots\}$ and $m_j \in \{-j, (-j+1), \ldots, (j-1), j\}$	(See above for terminology.)

Derivation using ladder operators

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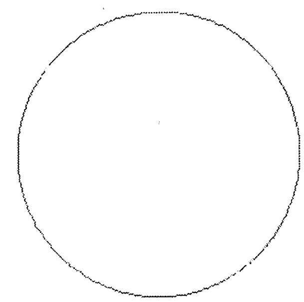
A common way to derive the quantization rules above is the method of *ladder operators*.[9] The ladder operators are defined:

$$J_+ \equiv J_x + iJ_y,$$
$$J_- \equiv J_x - iJ_y$$

Suppose a state $|\psi\rangle$ is a state in the simultaneous eigenbasis of J^2 and J_z (i.e., a state with a single, definite value of J^2 and a single, definite value of J_z). Then using the commutation relations, one can prove that $J_+|\psi\rangle$ and $J_-|\psi\rangle$ are *also* in the simultaneous eigenbasis, with the same value of J^2, but where $J_z|\psi\rangle$ is increased or decreased by $\hbar$, respectively. (It is also possible that one or both of these result vectors is the zero vector.) (For a proof, see ladder operator#Angular momentum.)

By manipulating these ladder operators and using the commutation rules, it is possible to prove almost all of the quantization rules above.



In this standing wave on a circular string, the circle is broken into exactly 8 wavelengths. A standing wave like this can have 0, 1, 2, or any integer number of wavelengths around the circle, but it *cannot* have a non-integer number of wavelengths like 8.3. In quantum mechanics, angular momentum is quantized for a similar reason.

Click [show] on the right to see more details in the ladder-operator proof of the quantization rules[9]

Before starting the main proof, we will note a useful fact: That J_x^2, J_y^2, J_z^2 are positive-semidefinite operators, meaning that all their eigenvalues are nonnegative. That also implies that the same is true for their sums, including $J^2 = J_x^2 + J_y^2 + J_z^2$ and $\left(J^2 - J_z^2\right) = \left(J_x^2 + J_y^2\right)$. The reason is that the square of *any* Hermitian operator is always positive semidefinite. (A Hermitian operator has real eigenvalues, so the squares of those eigenvalues are nonnegative.)

As above, assume that a state $|\psi\rangle$ is a state in the simultaneous eigenbasis of J^2 and J_z. Its eigenvalue with respect to J^2 can be written in the form $\hbar^2 j(j+1)$ for some real number $j > 0$ (because as mentioned in the previous paragraph, J^2 has nonnegative eigenvalues), and its eigenvalue with respect to J_z can be written $\hbar m$ for some real number *m*. Instead of $|\psi\rangle$ we will use the more descriptive notation $|\psi\rangle = |j, m\rangle$.

Next, consider the sequence ("ladder") of states

$$\{\ldots\ , \ J_-J_-|j,m\rangle,\ J_-|j,m\rangle,\ |j,m\rangle,\ J_+|j,m\rangle,\ J_+J_+|j,m\rangle,\ \ldots\}$$

Some entries in this infinite sequence may be the zero vector (as we will see). However, as described above, all the nonzero entries have the same value of J^2, and among the nonzero entries, each entry has a value of J_z which is exactly $\hbar$ more than the previous entry.

In this ladder, there can only be a finite number of nonzero entries, with infinite copies of the zero vector on the left and right. The reason is, as mentioned above, $(J^2 - J_z^2)$ is positive-semidefinite, so if any quantum state is an eigenvector of both J^2 and J_z^2, the former eigenvalue is larger. The states in

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the ladder all have the same J^2 eigenvalue, but going very far to the left or the right, the J_z^2 eigenvalue gets larger and larger. The only possible resolution is, as mentioned, that there are only finitely many nonzero entries in the ladder.

Now, consider the last nonzero entry to the right of the ladder, $|j, m_{max}\rangle$. This state has the property that $J_+|j, m_{max}\rangle = 0$. As proven in the ladder operator article,

$$J_+|j,m\rangle = \hbar\sqrt{j(j+1) - m(m+1)}|j, m+1\rangle$$

If this is zero, then $j(j+1) = m_{\max}(m_{\max}+1)$, so $j = m$ or $j = -m-1$. However, because $J^2 - J_z^2$ is positive-semidefinite, $\hbar^2 j(j+1) \geq (\hbar m)^2$, which means that the only possibility is $m_{\max} = j$.

Similarly, consider the first nonzero entry on the left of the ladder, $|j, m_{\min}\rangle$. This state has the property that $J_-|j, m_{\min}\rangle = 0$. As proven in the ladder operator article,

$$J_-|j,m\rangle = \hbar\sqrt{j(j+1) - m(m-1)}|j, m-1\rangle$$

As above, the only possibility is that $m_{\min} = -j$

Since *m* changes by 1 on each step of the ladder, $(j-(-j))$ is an integer, so *j* is an integer or half-integer (0 or 0.5 or 1 or 1.5...).

Since **S** and **L** have the same commutation relations as **J**, the same ladder analysis works for them.

The ladder-operator analysis does ***not*** explain one aspect of the quantization rules above: the fact that **L** (unlike **J** and **S**) cannot have half-integer quantum numbers. This fact can be proven (at least in the special case of one particle) by writing down every possible eigenfunction of L^2 and L_z, (they are the spherical harmonics), and seeing explicitly that none of them have half-integer quantum numbers.[10] An alternative derivation is below.

Visual interpretation

Since the angular momenta are quantum operators, they cannot be drawn as vectors like in classical mechanics. Nevertheless, it is common to depict them heuristically in this way. Depicted on the right is a set of states with quantum numbers $\ell = 2$, and $m_\ell = -2, -1, 0, 1, 2$ for the five cones from bottom to top. Since $|L| = \sqrt{L^2} = \hbar\sqrt{6}$, the vectors are all shown with length $\hbar\sqrt{6}$. The rings represent the fact that L_z is known with certainty, but L_x and L_y are unknown; therefore every classical vector with the appropriate length and *z*-component is drawn, forming a cone. The expected value of the angular momentum for a given ensemble of systems in the quantum state characterized by ℓ and m_ℓ could be somewhere on this cone while it cannot be defined for a single system (since the components of L do not commute with each other).

Quantization in macroscopic systems

The quantization rules are technically true even for macroscopic systems, like the angular momentum **L**

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of a spinning tire. However they have no observable effect. For example, if $L_z/\hbar$ is roughly 100000000, it makes essentially no difference whether the precise value is an integer like 100000000 or 100000001, or a non-integer like 100000000.2—the discrete steps are too small to notice.

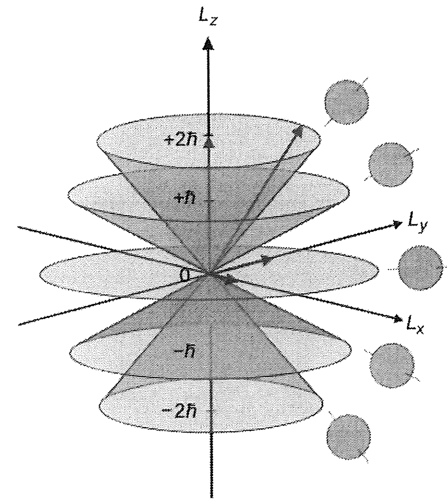


Illustration of the vector model of orbital angular momentum.

Angular momentum as the generator of rotations

The most general and fundamental definition of angular momentum is as the *generator* of rotations.[5] More specifically, let $R(\hat{n},\phi)$ be a rotation operator, which rotates any quantum state about axis $\hat{n}$ by angle ϕ. As $\phi \to 0$, the operator $R(\hat{n},\phi)$ approaches the identity operator, because a rotation of 0° maps all states to themselves. Then the angular momentum operator $J_{\hat{n}}$ about axis $\hat{n}$ is defined as:[5]

$$J_{\hat{n}} \equiv i\hbar \lim_{\phi\to 0} \frac{R(\hat{n},\phi)-1}{\phi} = i\hbar \left.\frac{\partial R(\hat{n},\phi)}{\partial\phi}\right|_{\phi=0}$$

where 1 is the identity operator. Also notice that R is an additive morphism : $R(\hat{n},\phi_1+\phi_2) = R(\hat{n},\phi_1)\,R(\hat{n},\phi_2)$; as a consequence[5]

$$R(\hat{n},\phi) = \exp\left(-\frac{i\phi J_{\hat{n}}}{\hbar}\right)$$

where exp is matrix exponential.

In simpler terms, the total angular momentum operator characterizes how a quantum system is changed when it is rotated. The relationship between angular momentum operators and rotation operators is the same as the relationship between Lie algebras and Lie groups in mathematics, as discussed further below.

Just as **J** is the generator for rotation operators, **L** and **S** are generators for modified partial rotation operators. The operator

$$R_{\text{spatial}}(\hat{n},\phi) = \exp\left(-\frac{i\phi L_{\hat{n}}}{\hbar}\right),$$

rotates the position (in space) of all particles and fields, without rotating the internal (spin) state of any particle. Likewise, the operator

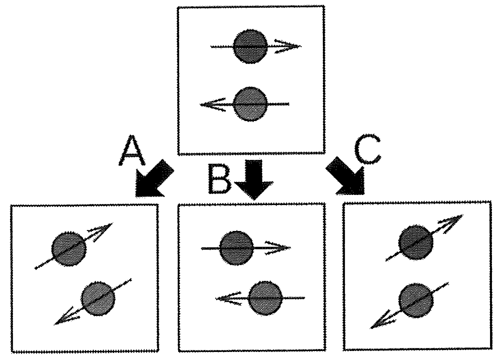
$$R_{\text{internal}}(\hat{n},\phi) = \exp\left(-\frac{i\phi S_{\hat{n}}}{\hbar}\right),$$

rotates the internal (spin) state of all particles, without moving any particles or fields in space. The relation **J** = **L** + **S** comes from:

$$R(\hat{n},\phi) = R_{\text{internal}}(\hat{n},\phi)\,R_{\text{spatial}}(\hat{n},\phi)$$

i.e. if the positions are rotated, and then the internal states are rotated, then altogether the complete system has been rotated.

SU(2), SO(3), and 360° rotations



The different types of rotation operators. The top box shows two particles, with spin states indicated schematically by the arrows.

A. The operator R, related to **J**, rotates the entire system.
B. The operator R_{spatial}, related to **L**, rotates the particle positions without altering their internal spin states.
C. The operator R_{internal}, related to **S**, rotates the particles' internal spin states without changing their positions.

Although one might expect $R(\hat{n}, 360°) = 1$ (a rotation of 360° is the identity operator), this is *not* assumed in quantum mechanics, and it turns out it is often not true: When the total angular momentum quantum number is a half-integer (1/2, 3/2, etc.), $R(\hat{n}, 360°) = -1$, and when it is an integer, $R(\hat{n}, 360°) = +1$.[5] Mathematically, the structure of rotations in the universe is *not* SO(3), the group of three-dimensional rotations in classical mechanics. Instead, it is SU(2), which is identical to SO(3) for small rotations, but where a 360° rotation is mathematically distinguished from a rotation of 0°. (A rotation of 720° is, however, the same as a rotation of 0°.)[5]

On the other hand, $R_{\text{spatial}}(\hat{n}, 360°) = +1$ in all circumstances, because a 360° rotation of a *spatial* configuration is the same as no rotation at all. (This is different from a 360° rotation of the *internal* (spin) state of the particle, which might or might not be the same as no rotation at all.) In other words, the R_{spatial} operators carry the structure of SO(3), while R and R_{internal} carry the structure of SU(2).

From the equation $+1 = R_{\text{spatial}}(\hat{z}, 360°) = \exp(-2\pi i L_z/\hbar)$, one picks an eigenstate $L_z|\psi\rangle = m\hbar|\psi\rangle$ and draws

$$e^{-2\pi i m} = 1$$

which is to say that the orbital angular momentum quantum numbers can only be integers, not half-integers.

Connection to representation theory

Starting with a certain quantum state $|\psi_0\rangle$, consider the set of states $R(\hat{n}, \phi)|\psi_0\rangle$ for all possible $\hat{n}$ and ϕ, i.e. the set of states that come about from rotating the starting state in every possible way. This is a vector space, and therefore the manner in which the rotation operators map one state onto another is a *representation* of the group of rotation operators.

> *When rotation operators act on quantum states, it forms a representation of the Lie group SU(2) (for R and R_{internal}), or SO(3) (for R_{spatial}).*

From the relation between **J** and rotation operators,

> *When angular momentum operators act on quantum states, it forms a representation of the Lie algebra $\mathfrak{su}(2)$ or $\mathfrak{so}(3)$.*

(The Lie algebras of SU(2) and SO(3) are identical.)

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The ladder operator derivation above is a method for classifying the representations of the Lie algebra SU(2).

Connection to commutation relations

Classical rotations do not commute with each other: For example, rotating 1° about the x-axis then 1° about the y-axis gives a slightly different overall rotation than rotating 1° about the y-axis then 1° about the x-axis. By carefully analyzing this noncommutativity, the commutation relations of the angular momentum operators can be derived.[5]

(This same calculational procedure is one way to answer the mathematical question "What is the Lie algebra of the Lie groups SO(3) or SU(2)?")

Conservation of angular momentum

The Hamiltonian H represents the energy and dynamics of the system. In a spherically-symmetric situation, the Hamiltonian is invariant under rotations:

$$RHR^{-1} = H$$

where R is a rotation operator. As a consequence, $[H, R] = 0$, and then $[H, \mathbf{J}] = \mathbf{0}$ due to the relationship between $\mathbf{J}$ and R. By the Ehrenfest theorem, it follows that $\mathbf{J}$ is conserved.

To summarize, if H is rotationally-invariant (spherically symmetric), then total angular momentum $\mathbf{J}$ is conserved. This is an example of Noether's theorem.

If H is just the Hamiltonian for one particle, the total angular momentum of that one particle is conserved when the particle is in a central potential (i.e., when the potential energy function depends only on $|\mathbf{r}|$). Alternatively, H may be the Hamiltonian of all particles and fields in the universe, and then H is *always* rotationally-invariant, as the fundamental laws of physics of the universe are the same regardless of orientation. This is the basis for saying conservation of angular momentum is a general principle of physics.

For a particle without spin, $\mathbf{J} = \mathbf{L}$, so orbital angular momentum is conserved in the same circumstances. When the spin is nonzero, the spin-orbit interaction allows angular momentum to transfer from $\mathbf{L}$ to $\mathbf{S}$ or back. Therefore, $\mathbf{L}$ is not, on its own, conserved.

Angular momentum coupling

Often, two or more sorts of angular momentum interact with each other, so that angular momentum can transfer from one to the other. For example, in spin-orbit coupling, angular momentum can transfer between $\mathbf{L}$ and $\mathbf{S}$, but only the total $\mathbf{J} = \mathbf{L} + \mathbf{S}$ is conserved. In another example, in an atom with two electrons, each has its own angular momentum $\mathbf{J}_1$ and $\mathbf{J}_2$, but only the total $\mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2$ is conserved.

In these situations, it is often useful to know the relationship between, on the one hand, states where $(J_1)_z, (J_1)^2, (J_2)_z, (J_2)^2$ all have definite values, and on the other hand, states where $(J_1)^2, (J_2)^2, J^2, J_z$ all have definite values, as the latter four are usually conserved (constants of motion). The procedure to go back and forth between these bases is to use Clebsch–Gordan coefficients.

One important result in this field is that a relationship between the quantum numbers for $(J_1)^2, (J_2)^2, J^2$:

$$j \in \{|j_1 - j_2|, (|j_1 - j_2| + 1), \ldots, (j_1 + j_2)\}.$$

For an atom or molecule with $\mathbf{J} = \mathbf{L} + \mathbf{S}$, the term symbol gives the quantum numbers associated with the operators L^2, S^2, J^2.

Orbital angular momentum in spherical coordinates

Angular momentum operators usually occur when solving a problem with spherical symmetry in spherical coordinates. The angular momentum in the spatial representation is[11][12]

$$\begin{aligned}
\mathbf{L} &= i\hbar\left(\frac{\hat{\boldsymbol{\theta}}}{\sin(\theta)}\frac{\partial}{\partial\phi} - \hat{\boldsymbol{\phi}}\frac{\partial}{\partial\theta}\right)\\
&= i\hbar\left(\hat{\mathbf{x}}\left(\sin(\phi)\frac{\partial}{\partial\theta} + \cot(\theta)\cos(\phi)\frac{\partial}{\partial\phi}\right) + \hat{\mathbf{y}}\left(-\cos(\phi)\frac{\partial}{\partial\theta} + \cot(\theta)\sin(\phi)\frac{\partial}{\partial\phi}\right) - \hat{\mathbf{z}}\frac{\partial}{\partial\phi}\right)\\
L_+ &= \hbar e^{i\phi}\left(\frac{\partial}{\partial\theta} + i\cot(\theta)\frac{\partial}{\partial\phi}\right),\\
L_- &= \hbar e^{-i\phi}\left(-\frac{\partial}{\partial\theta} + i\cot(\theta)\frac{\partial}{\partial\phi}\right),\\
L^2 &= -\hbar^2\left(\frac{1}{\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2(\theta)}\frac{\partial^2}{\partial\phi^2}\right),\\
L_z &= -i\hbar\frac{\partial}{\partial\phi}.
\end{aligned}$$

In spherical coordinates the angular part of the Laplace operator can be expressed by the angular momentum. This leads to the relation

$$\Delta = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{L^2}{\hbar^2 r^2}.$$

When solving to find eigenstates of the operator L^2, we obtain the following

$$\begin{aligned}
L^2|l,m\rangle &= \hbar^2 l(l+1)|l,m\rangle\\
L_z|l,m\rangle &= \hbar m|l,m\rangle
\end{aligned}$$

where

$$\langle\theta,\phi|l,m\rangle = Y_{l,m}(\theta,\phi)$$

are the spherical harmonics.[13]

See also

- Runge–Lenz vector (used to describe the shape and orientation of bodies in orbit)
- Holstein–Primakoff transformation
- Jordan map (Schwinger's bosonic model of angular momentum[14])
- Vector model of the atom
- Pauli–Lubanski pseudovector
- Angular momentum diagrams (quantum mechanics)

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- Spherical basis
- Tensor operator
- Orbital magnetization
- Orbital angular momentum of free electrons
- Orbital angular momentum of light

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Ladder operator

In linear algebra (and its application to quantum mechanics), a **raising** or **lowering operator** (collectively known as **ladder operators**) is an operator that increases or decreases the eigenvalue of another operator. In quantum mechanics, the raising operator is sometimes called the creation operator, and the lowering operator the annihilation operator. Well-known applications of ladder operators in quantum mechanics are in the formalisms of the quantum harmonic oscillator and angular momentum.

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Terminology

There is some confusion regarding the relationship between the raising and lowering ladder operators and the creation and annihilation operators commonly used in quantum field theory. The creation operator $a_i^\dagger$ increments the number of particles in state *i*, while the corresponding annihilation operator a_i decrements the number of particles in state *i*. This clearly satisfies the requirements of the above definition of a ladder operator: the incrementing or decrementing of the eigenvalue of another operator (in this case the particle number operator).

Confusion arises because the term *ladder operator* is typically used to describe an operator that acts to increment or decrement a quantum number describing the state of a system. To change the state of a particle with the creation/annihilation operators of QFT requires the use of *both* an annihilation operator to remove a particle from the initial state *and* a creation operator to add a particle to the final state.

The term "ladder operator" is also sometimes used in mathematics, in the context of the theory of Lie algebras and in particular the affine Lie algebras, to describe the su(2)

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subalgebras, from which the root system and the highest weight modules can be constructed by means of the ladder operators.[1] In particular, the highest weight is annihilated by the raising operators; the rest of the positive root space is obtained by repeatedly applying the lowering operators (one set of ladder operators per subalgebra).

General formulation

Suppose that two operators X and N have the commutation relation,

$$[N, X] = cX,$$

for some scalar c. If $|n\rangle$ is an eigenstate of N with eigenvalue equation,

$$N|n\rangle = n|n\rangle,$$

then the operator X acts on $|n\rangle$ in such a way as to shift the eigenvalue by c:

$$\begin{aligned} NX|n\rangle &= (XN + [N, X])|n\rangle \\ &= XN|n\rangle + [N, X]|n\rangle \\ &= Xn|n\rangle + cX|n\rangle \\ &= (n + c)X|n\rangle. \end{aligned}$$

In other words, if $|n\rangle$ is an eigenstate of N with eigenvalue n then $X|n\rangle$ is an eigenstate of N with eigenvalue $n + c$ or it is zero. The operator X is a *raising operator* for N if c is real and positive, and a *lowering operator* for N if c is real and negative.

If N is a Hermitian operator then c must be real and the Hermitian adjoint of X obeys the commutation relation:

$$[N, X^\dagger] = -cX^\dagger.$$

In particular, if X is a lowering operator for N then $X^\dagger$ is a raising operator for N and vice versa.

Angular momentum

A particular application of the ladder operator concept is found in the quantum mechanical treatment of angular momentum. For a general angular momentum vector, **J**, with components, J_x, J_y and J_z we define the two ladder operators, J_+ and J_-:[2]

$$J_+ = J_x + iJ_y,$$

$$J_- = J_x - iJ_y,$$

where i is the imaginary unit.

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The commutation relation between the cartesian components of *any* angular momentum operator is given by

$$[J_i, J_j] = i\hbar\epsilon_{ijk}J_k,$$

where ε_{ijk} is the Levi-Civita symbol and each of i, j and k can take any of the values x, y and z. From this the commutation relations between the ladder operators and J_z can easily be obtained:

$$[J_z, J_\pm] = \pm\hbar J_\pm.$$
$$[J_+, J_-] = 2\hbar J_z.$$

The properties of the ladder operators can be determined by observing how they modify the action of the J_z operator on a given state:

$$\begin{aligned} J_z J_\pm |j\,m\rangle &= (J_\pm J_z + [J_z, J_\pm])\,|j\,m\rangle \\ &= (J_\pm J_z \pm \hbar J_\pm)\,|j\,m\rangle \\ &= \hbar\,(m \pm 1)\,J_\pm |j\,m\rangle. \end{aligned}$$

Compare this result with:

$$J_z|j\,(m \pm 1)\rangle = \hbar(m \pm 1)|j\,(m \pm 1)\rangle.$$

Thus we conclude that $J_\pm|j\,m\rangle$ is some scalar multiplied by $|j\,m{\pm}1\rangle$,

$$J_+|j\,m\rangle = \alpha|j\,m+1\rangle,$$

$$J_-|j\,m\rangle = \beta|j\,m-1\rangle.$$

This illustrates the defining feature of ladder operators in quantum mechanics: the incrementing (or decrementing) of a quantum number, thus mapping one quantum state onto another. This is the reason that they are often known as raising and lowering operators.

To obtain the values of α and β we first take the norm of each operator, recognizing that J_+ and J_- are a Hermitian conjugate pair ($J_\pm = J_\mp^\dagger$),

$$\langle j\,m|J_+^\dagger J_+|j\,m\rangle = \langle j\,m|J_- J_+|j\,m\rangle = \langle j\,(m+1)|\alpha^*\alpha|j\,(m+1)\rangle = |\alpha|^2\,,$$

$$\langle j\,m|J_-^\dagger J_-|j\,m\rangle = \langle j\,m|J_+ J_-|j\,m\rangle = \langle j\,(m-1)|\beta^*\beta|j\,(m-1)\rangle = |\beta|^2\,.$$

The product of the ladder operators can be expressed in terms of the commuting pair J^2 and J_z,

$$J_- J_+ = (J_x - iJ_y)(J_x + iJ_y) = J_x^2 + J_y^2 + i[J_x, J_y] = J^2 - J_z^2 - \hbar J_z,$$

$$J_+J_- = (J_x + iJ_y)(J_x - iJ_y) = J_x^2 + J_y^2 - i[J_x, J_y] = J^2 - J_z^2 + \hbar J_z.$$

Thus we can express the values of $|\alpha|^2$ and $|\beta|^2$ in terms of the eigenvalues of J^2 and J_z,

$$|\alpha|^2 = \hbar^2 j(j+1) - \hbar^2 m^2 - \hbar^2 m = \hbar^2 (j-m)(j+m+1),$$

$$|\beta|^2 = \hbar^2 j(j+1) - \hbar^2 m^2 + \hbar^2 m = \hbar^2 (j+m)(j-m+1).$$

The phases of α and β are not physically significant, thus they can be chosen to be positive and real (Condon-Shortley phase convention). We then have:[3]

$$J_+|j\,m\rangle = \hbar\sqrt{(j-m)(j+m+1)}|j\,m+1\rangle = \hbar\sqrt{j(j+1)-m(m+1)}|j\,m+1\rangle,$$

$$J_-|j\,m\rangle = \hbar\sqrt{(j+m)(j-m+1)}|j\,m-1\rangle = \hbar\sqrt{j(j+1)-m(m-1)}|j\,m-1\rangle.$$

Confirming that m is bounded by the value of j ($-j \le m \le j$) we have:

$$J_+|j\,j\rangle = 0,$$

$$J_-|j\,(-j)\rangle = 0.$$

The above demonstration is effectively the construction of the Clebsch-Gordan coefficients.

Applications in atomic and molecular physics

Many terms in the Hamiltonians of atomic or molecular systems involve the scalar product of angular momentum operators. An example is the magnetic dipole term in the hyperfine Hamiltonian,[4]

$$\hat{H}_{\mathrm{D}} = \hat{A}\mathbf{I}\cdot\mathbf{J},$$

where I is the nuclear spin. Angular momentum algebra can often be simplified by recasting it in the spherical basis. Using the notation of spherical tensor operators, the "-1", "0" and "+1" components of $\mathbf{J}^{(1)} \equiv \mathbf{J}$ are given by,[5]

$$J_{-1}^{(1)} = \frac{1}{\sqrt{2}}(J_x - iJ_y) = \frac{J_-}{\sqrt{2}}$$
$$J_0^{(1)} = J_z$$
$$J_{+1}^{(1)} = -\frac{1}{\sqrt{2}}(J_x + iJ_y) = -\frac{J_+}{\sqrt{2}}.$$

From these definitions it can be shown that the above scalar product can be expanded as

$$\mathbf{I}^{(1)} \cdot \mathbf{J}^{(1)} = \sum_{n=-1}^{+1} (-1)^n I_n^{(1)} J_{-n}^{(1)} = I_0^{(1)} J_0^{(1)} - I_{-1}^{(1)} J_{+1}^{(1)} - I_{+1}^{(1)} J_{-1}^{(1)},$$

The significance of this expansion is that it clearly indicates which states are coupled by this term in the Hamiltonian, that is those with quantum numbers differing by $m_i = \pm 1$ and $m_j = \mp 1$ *only*.

Harmonic oscillator

Another application of the ladder operator concept is found in the quantum mechanical treatment of the harmonic oscillator. We can define the lowering and raising operators as

$$a = \sqrt{\frac{m\omega}{2\hbar}} \left(\hat{x} + \frac{i}{m\omega} \hat{p} \right)$$
$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}} \left(\hat{x} - \frac{i}{m\omega} \hat{p} \right)$$

They provide a convenient means to extract energy eigenvalues without directly solving the system's differential equation.

History

Many sources credit Dirac with the invention of ladder operators.[6] Dirac's use of the ladder operators shows that the total angular momentum quantum number j needs to be a non-negative *half* integer multiple of ħ.

See also

- Creation and annihilation operators
- Quantum harmonic oscillator

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